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Approximation algorithms for terrain guarding ☆

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Abstract

We present approximation algorithms and heuristics for several variations of terrain guarding problems, where we need to guard a terrain in its entirety by a minimum number of guards. Terrain guarding has applications in telecommunications, namely in the setting up of antenna networks for wireless communication. Our approximation algorithms transform the terrain guarding instance into a MINIMUM SET COVER instance, which is then solved by the standard greedy approximation algorithm [J. Comput. System Sci. 9 (1974) 256–278]. The approximation algorithms achieve approximation ratios of $O(\log n)$, where n is the number of vertices in the input terrain. We also briefly discuss some heuristic approaches for solving other variations of terrain guarding problems, for which no approximation algorithms are known. These heuristic approaches do not guarantee non-trivial approximation ratios but may still yield good solutions. © 2002 Published by Elsevier Science B.V.

Keywords: Approximation algorithms; Computational complexity; Computational geometry; Terrain guarding

1. Introduction and problem definition

The cost of a telecommunications network largely depends on the number of nodes. Typically, a network for wireless communication consists of transmission stations (antennas) that receive and send signals. The set of antennas needs to cover a specific geographic region in its entirety. Putting up antennas is very costly, and hence telecommunication companies — among other objectives — aim at placing a minimum number of antennas that cover a given region such that communication between any two points in a given region is possible. While the traditional way of erecting antenna towers on the ground is still most-widely used, a novel approach is to put antennas up in the air: Balloons float at a certain fixed height and are held in geo-stationary position.

Communication between two points is possible, if at each point a mobile transmitter (such as a cellular phone) can communicate with an antenna in the network. Thus, in our abstract problem, each point in the region must be *covered* by at least one antenna. Communication between antennas and mobile transmitters is by means of electromagnetic wave propagation at high frequencies. Current frequencies are 900 and 1800 MHz in Europe and 1900 MHz in the US and the trend points towards frequencies even higher in the GHz-range. A straight *line-of-sight* approach models reality with sufficient precision in these frequency ranges, since the effects of reflection and refraction become negligible. Thus, we require each point in the region to be *visible* from at least one antenna in the network.

☆ The results in this paper are part of the author's PhD thesis [5].

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Since visibility between two points in a region is determined by the topology of the region to be covered, the geometric model used to model the region becomes important. In the geometric model, we call the region to be covered a *terrain*, which is described as a finite set of points in the plane together with a triangulation, and a height value associated with each point (this is also called a *triangulated irregular network* (TIN), see, e.g., [12]). Visibility on a terrain is defined on the basis of straight *lines-of-sight*: Two points above the terrain are mutually visible if their connecting straight line segment runs entirely above (or on) the terrain.

It turns out that the terrain covering problem can be seen to belong to quite a large family of geometric covering and guarding problems that have been studied for more than two decades. Victor Klee started the study by posing the following problem, which today is known as the *original art gallery problem*: How many guards are needed to see every point in the interior of an art gallery? In the abstract version of this problem, the input is a simple polygon with or without holes in the plane, representing the floor plan of the art gallery, and visibility is of course limited to the interior of the polygon. A simple polygon with or without holes is given by its ordered sequence of vertices on the outer boundary, together with an ordered sequence of vertices for each hole (if it contains holes). Two points in the polygone *see* each other if the straight line segment connecting the two points does not intersect the exterior (and the holes) of the polygon.

We now formally define the problems we are studying.

Definition 1. Let T be a simple polygon with holes. The problem MINIMUM VERTEX (POINT) GUARD ON POLYGON is the problem of finding a minimum subset S of the set of vertices of T (a minimum set S of points in the interior of T) such that each point on the boundary and in the interior of T is visible from at least one point in S.

As usual, a minimum subset of a set denotes a subset of smallest cardinality among all candidate subsets.

Definition 2. Let T be a terrain. The problem MINIMUM VERTEX (POINT) GUARD ON TERRAIN is the problem of finding a minimum subset S of vertices of T (minimum set of points on T) such that every point on T is visible from a point in S. The points in S are called *guard points*.

Definition 3. Let T be a terrain, and let h be a height value, such that the plane $z = h$ lies entirely above (or partially on) T. The problem MINIMUM FIXED HEIGHT GUARD ON TERRAIN is the problem of finding a minimum set S of points in space at height h such that every point on T is visible from a point in S.

We define additional variations, which will allow us to cut running times significantly, while still yielding good solutions. These variations have the additional restriction that each triangle in the triangulation of T must be visible from a single point in the guard set S; that is, guards are not allowed to cooperatively see a triangle in T's triangulation, contrary to the problem versions above. We denote these problem versions by MINIMUM VERTEX/POINT/FIXED HEIGHT GUARD ON TERRAIN WITH TRIANGLE RESTRICTION. In most practical applications, the triangles of the triangulation of a given terrain are very small compared to the overall size of the terrain. Thus, a single guard is very likely to cover a large number of triangles completely and only a small number only partially. It, therefore, seems reasonable to ignore the partially covered triangles, if we achieve a significant reduction in running time.

In our approximation algorithms, we first express the terrain guarding problem as an instance of MINIMUM SET COVER, which is defined as follows:

Definition 4. Let $E = \{e_1, \ldots, e_n\}$ be a finite set (called universe) of elements, and let $S = \{s_1, \ldots, s_m\}$ be a collection of subsets of E, i.e., $s_j \subseteq E$ for $1 \leqslant j \leqslant m$. The problem SET COVER is the problem of finding a minimum subset $S' \subseteq S$ such that every element $e_i \in E$, $1 \leqslant i \leqslant n$ belongs to at least one subset in S'. For ease of discussion, let the elements in E and the subsets in S have an arbitrary, but fixed order, denoted by the index.

A simple greedy algorithm for MINIMUM SET COVER, which consists of recursively adding to the solution a set that contains a maximum number of elements not yet covered by the current solution, achieves an approximation ratio of $\ln n + 1$ [11]. It turns out that this approximation algorithm is the best possible (up to low-order terms) [1,9].

There are several surveys on art galleries [14–16]. Many upper and lower bounds on the number of necessary guards are known for different guarding problems, while comparatively few results are known on the computational complexity of these guarding problems.

It is known that guarding a given polygon is NP-hard in all versions, i.e., MINIMUM VERTEX (POINT) GUARD ON POLYGON is NP-hard [13]. Moreover, MINIMUM VERTEX (POINT) GUARD ON POLYGON is APX-hard [8] for polygons without holes, which means that there exists a constant $\varepsilon > 0$ such that no polynomial time algorithm can achieve an approximation ratio of $1 + \varepsilon$. MINIMUM VERTEX (POINT) GUARD ON POLYGON as well as MINIMUM FIXED HEIGHT/VERTEX/POINT GUARD ON TERRAIN (WITH TRIANGLE RESTRICTION) cannot be approximated with an approximation ratio of $O(\log n)$ [5] (or [7,6]). An approximation algorithms for MINIMUM VERTEX GUARD ON POLYGON, which achieves an approximation ratio of $O(\log n)$, is also known [10].

In this report, we propose approximation algorithms for MINIMUM VERTEX GUARD ON TERRAIN (WITH TRIANGLE RESTRICTION) and for MINIMUM FIXED HEIGHT/POINT GUARD ON TERRAIN WITH TRIANGLE RESTRICTION, which achieve logarithmic approximation ratios. These approximation algorithms are thus optimal up to a constant factor. All our approximation algorithms first transform the guarding problem into a MINIMUM SET COVER problem, which is then solved by the simple greedy approximation algorithm mentioned above.

For the remaining terrain guarding problem, i.e., MINIMUM FIXED HEIGHT/POINT GUARD ON TERRAIN as well as for MINIMUM POINT GUARD ON POLYGON, we briefly outline some heuristic approaches, which are not guaranteed to achieve any non-trivial approximation ratios, but may still be very useful and good in practice.

This report is structured as follows: In Section 2 we repeat the concept for the approximation algorithm for guarding polygons as proposed in [10]. We generalize the concept of the algorithm for polygons to terrains and obtain our approximation algorithms for terrain guarding problems in Section 3. Section 4 contains heuristic approaches for the problems discussed. Section 5 contains concluding remarks.

2. Approximating polygon guarding problems

It is known that the problem MINIMUM VERTEX GUARD ON POLYGON is approximable with a ratio of $O(\log n)$, where n is the number of polygon vertices [10]. As we will employ similar concepts for our algorithms for guarding terrains, we briefly outline this algorithm.

The approximation algorithm partitions the interior of the input polygon into "basic" convex components that are either completely visible or invisible from any vertex guard. These basic convex components are obtained by drawing lines through all pairs of vertices of the polygon. Each of the $O(n^2)$ lines intersects at most $O(n^2)$ lines, which gives a total of $O(n^4)$ intersection points. Each intersection point is the vertex of a convex component that is minimum (in the convex component) with respect to the y-axis (where the y-axis is arbitrary but fixed). Therefore, we have $O(n^4)$ basic convex components.

The problem is then transformed into an instance of MINIMUM SET COVER with an element for each convex component and a set for each polygon vertex (or polygon edge). The sets contain as elements exactly the convex components that are visible from the vertex.

MINIMUM SET COVER can be approximated with an approximation ratio that is logarithmic in the number of elements of the MINIMUM SET COVER instance. Since we have a polynomial number of elements, the approximation ratio that the greedy algorithm achieves remains logarithmic.

3. Approximating terrain guarding problems

We generalize the notion of basic convex components introduced in Section 2 to terrains. This will help us obtain approximation algorithms for several terrain guarding problems. We obtain convex components by constructing planes through vertices and line segments of the terrain. More precisely, let v_i, v_j and v_k be vertices in a terrain T, where the vertices v_i and v_j are neighbors, i.e., the two vertices are connected by a line segment in the triangulation of the terrain. For each line segment v_i, v_j and each vertex v_k on the terrain, we construct a plane that contains the line segment from v_i to v_j and the vertex v_k. Since there are only $O(n)$ line segments in the triangulation of T, this gives a total of $O(n^2)$ planes. These planes partition space into three-dimensional *cells*, which in turn contain two-dimensional *faces* that are defined by (one-dimensional) *points*, which are the intersection points of three planes. We call this partition the *arrangement* of T. The arrangement consists of $O(n^6)$ intersection points, faces, and cells that can be computed in time $O(n^6)$. [1]

Lemma 1. *Let C be a cell in the arrangement of a terrain T. Then, every point in the interior of cell C (i.e., any point not on the faces or intersection points that also belong to C) sees exactly the same vertices of terrain T as any other point in the interior of C.*

Proof. Let a, b be two points in the interior of C and let v_i be a vertex on the terrain T. Assume by contradiction that b sees v_i, while a does not see v_i. Then, let a' be the intersection point of the terrain T and the line segment from point a to vertex v_i that is closest to point a. Since a does not see v_i, there always must be such a point a' that blocks the view. Let b' be the intersection point of the line segment from b to v_i and the plane defined by the triangle of the terrain, on which point a' lies. Since b sees v_i, b' cannot lie on the terrain, and in particular it cannot lie on the same triangle as a'. Consider the plane through vertex v_i and the line segment of the terrain that intersects with the line segment from a' to b'. This plane is a part of the arrangement, but it cuts cell C apart, as it separates points a and b from each other. Therefore, cell C is not a cell of the arrangement. □

Lemma 2. *Let C be a cell in the arrangement of a terrain T. Then, every point in the interior of cell C sees exactly the same line segments of the triangulation of terrain T completely as any other point in the interior of C.*

Proof. Let a, b be two points in the interior of C and let c be the line segment in the triangulation of T with vertices v_i and v_j as endpoints. Assume by contradiction that b sees c, while a does not see c completely. Since a and b are both in cell C we can assume by Lemma 1 that they both see the two vertices v_i and v_j. Consider the points on the line segment from point a to point b as we move from a towards b. Since a does not see segment c completely, but b does, there must be a point p from where c is completely visible for the first time (as we move along the line segment from a to b). Remember that by Lemma 1 vertices v_i and v_j are always visible as we move. Therefore, by definition of point p, there must be a terrain vertex v_k (other that v_i and v_j) on the plane defined by v_i, v_j and p. Otherwise, p would not be the first point to completely see segment c, or not all points would see vertices v_i and v_j. The plane defined by v_i, v_j and p is equal to the plane defined by v_i, v_j and v_k. This plane, however, is part of the arrangement, since v_i and v_j are neighbors in the triangulation of terrain T; it separates points a and b, and therefore, C is not a cell of the arrangement. □

Lemma 3. *Let F be a face in the arrangement of a terrain T. Then, every point in the interior of face F (i.e., points not on the boundary of F) sees exactly the same vertices and the same complete line segments of the triangulation of terrain T as any other point in the interior of F.*

Proof. The proofs for the vertices and the line segments are the same as the proofs for Lemmas 1 and 2, respectively. □

Before we propose a first approximation algorithm, let us mention a few elementary facts:

- For each cell or face of the arrangement of a terrain T, the intersection points that are on the boundary of the cell or face see all the vertices and line segments in the triangulation of T that any other point in the interior of the cell or face sees. The intersection points may, however, see a few additional vertices and line segments.
- For any point p in space and any vertex v_i on terrain T, we determine if p sees v_i as follows: Compute all line segments in the triangulation of T that intersect the line from p to v_i in the orthogonal,

[1] These numbers are obtained easily using standard analysis of properties of arrangements that can be found in any textbook on computational geometry such as [2].

two-dimensional projection onto the x–y plane. Then, check for each such segment whether the line from p to v_i is above or below the segment (with respect to the z-axis). Point p sees vertex v_i, exactly if each segment is below the line from p to v_i. This can be computed in time $O(n)$.

- For any point p in space and any line segment in the triangulation of T with endpoints v_i and v_j, we determine if p completely sees the line segment as follows: We first determine whether p sees the two vertices v_i and v_j. If this is affirmative, we check for each vertex on the terrain, which lies in the triangle v_i, v_j, p in its orthogonal projection onto the x–y plane, whether the vertex lies above or below the triangle v_i, v_j, p (with respect to the z-axis). Point p completely sees the segment from v_i to v_j, exactly if each such vertex lies below the triangle. This can be computed in time $O(n)$.
- We can determine if a point in space completely sees a triangle on the terrain by determining if it completely sees all three sides of the triangle. This takes $O(n)$ time.

We are now ready to prove approximation results.

Theorem 1. MINIMUM FIXED HEIGHT GUARD ON TERRAIN WITH TRIANGLE RESTRICTION *can be approximated by a polynomial time algorithm with a ratio of* $O(\log n)$, *where* n *is the number of terrain vertices.*

Proof. Consider the intersection of the arrangement of terrain T with the plane at $z = h$. This intersection is itself a two-dimensional arrangement. Since we know from Lemmas 2 and 3 that all interior points in a cell or in a face see the same line segments in the triangulation of T and the boundary points may see a few additional line segments, it suffices to determine at each of the $O(n^6)$ intersection points (i.e., points in the two-dimensional arrangement at height h, where two lines cross), which triangles it sees completely. This can be done in $O(n^6 \cdot n \cdot n) = O(n^8)$ time.

We can now interpret this information as a MINIMUM SET COVER instance, where each triangle in the terrain is an element and each intersection point defines a set, namely the set of triangles that it sees completely. This instance consists of $O(n)$ elements and $O(n^6)$ different sets. We now solve the MINIMUM SET COVER instance approximately, by applying the greedy algorithm for MINIMUM SET COVER. The greedy algorithm runs in time $O(n^8)$ as does the whole approximation algorithm. It achieves an approximation ratio of $O(\log n)$. □

Theorem 2. MINIMUM VERTEX GUARD ON TERRAIN WITH TRIANGLE RESTRICTION *can be approximated by a polynomial time algorithm with a ratio of* $O(\log n)$, *where* n *is the number of terrain vertices.*

Proof. We again build a MINIMUM SET COVER instance, where each triangle in the terrain is an element and each vertex defines a set, i.e., the set of triangles that it sees completely. In order to do this, we have to compute n sets, each of which takes time $O(n^2)$. This gives a total construction time of $O(n^3)$.

Solving the MINIMUM SET COVER instance by applying the greedy algorithm takes time $O(n^3)$ and achieves an approximation ratio of $O(\log n)$. □

Theorem 3. MINIMUM VERTEX GUARD ON TERRAIN *can be approximated by a polynomial time algorithm with a ratio of* $O(\log n)$, *where* n *is the number of terrain vertices.*

Proof. Consider the intersection of the arrangement of T with the terrain T itself. This intersection partitions all triangles of the terrain into two-dimensional cells. Within such a cell, all points see the same set of vertices according to Lemma 3. The “inverse” holds as well: Any vertex in the terrain either sees such cell completely or not at all (except for points on the boundary of the cell). There are $O(n^6)$ such cells that can be computed in time $O(n^6)$. Note that we can determine in time $O(n)$ whether a vertex sees a cell, by testing whether it sees an interior point of the cell. [2]

We construct a MINIMUM SET COVER instance, where each cell in the terrain, which results from intersecting the arrangement of T with T itself, is an element and each vertex defines a set, i.e., the set of all cells that it sees completely. We have to compute n sets, each of which takes time $O(n^7)$. This gives a total construction time of $O(n^8)$.

We again solve the MINIMUM SET COVER instance by applying the greedy algorithm. This takes

[2] We can find an interior point by drawing two arbitrary diagonals and taking the intersection or — in the case of a triangle — by computing its center of gravity.

time $O(n^8)$ and achieves an approximation ratio of $O(\log n)$. □

Theorem 4. MINIMUM POINT GUARD ON TERRAIN WITH TRIANGLE RESTRICTION *can be approximated by a polynomial time algorithm with a ratio of* $O(\log n)$, *where* n *is the number of terrain vertices.*

Proof. Consider once again the intersection of the arrangement of T with the terrain T itself that partitions all triangles of the terrain into two-dimensional cells. Within such a cell, all points see the same set of line segments of the triangulation of T according to Lemma 3; they also all see the same set of triangles on the terrain. Therefore, it suffices to place point guards at intersection points (i.e., points in on the terrain, where two lines of the arrangement cross).

We construct a MINIMUM SET COVER instance, where each triangle in the triangulation of the terrain is an element and each intersection point on the terrain defines a set, namely the set of all triangles that it sees completely. We have to compute $O(n^6)$ sets, each of which takes time $O(n^2)$. This gives a total construction time of $O(n^8)$.

We again solve the MINIMUM SET COVER instance by applying the greedy algorithm. This takes time $O(n^8)$ and achieves an approximation ratio of $O(\log n)$. □

In all the approximation algorithms proposed for terrain guarding problems in the proofs of the previous lemmas, we have focused on the polynomiality of these algorithms. It is possible that the running times of our approximation algorithms may be improved by adopting algorithms that compute the horizon from a point in the terrain [4].

4. Heuristic approaches for guarding problems

Unfortunately, no sophisticated approximation algorithms are known for MINIMUM POINT GUARD ON POLYGON (TERRAIN) and MINIMUM FIXED HEIGHT GUARD ON TERRAIN. These problems seem to defy all attempts to somehow discretize the space of all possible guard positions. In fact, it is not even known whether the corresponding decision problems are in NP. However, we can come up with several approaches to find good solutions for these problems, even if these solutions are not provably good.

A trivial approximation algorithm for MINIMUM POINT GUARD ON POLYGON simply returns all n vertices as a (feasible) solution. This algorithm achieves an approximation ratio of n, because at least one guard is needed in each feasible solution. For input polygons without holes, we can improve this ratio slightly by applying an algorithm that places $\lfloor n/3 \rfloor$ guards that together see all of the interior of the polygon; this could be done in a similar way for input polygons with holes (see [16] for details). However, the approximation ratio remains $O(n)$. Corresponding trivial approximation algorithms for MINIMUM POINT (FIXED HEIGHT) GUARD ON TERRAIN simply place a guard at each vertex (above each vertex at height h). The resulting approximation ratios of n can be slightly improved by a constant factor by applying an algorithm [3] that always places $\lfloor 3n/5 \rfloor$ guards on a terrain that together see all of the terrain.[3] Alternatively, we can reduce the approximation ratio for MINIMUM FIXED HEIGHT GUARD ON TERRAIN to $n/2$ by determining whether height h is large enough such that the whole terrain can be seen from one single guard at some point at height h. The position of such a guard can be computed in linear time using linear programming (mentioned in [17] as the problem of computing the *lowest watch tower*). An approximation algorithm for MINIMUM FIXED HEIGHT GUARD ON TERRAIN could return the position of such a guard and, if no such guard exists, proceed with the trivial algorithm. However, the approximation ratios remain $O(n)$.

A better approach for MINIMUM POINT GUARD ON POLYGON returns the (suboptimum) solution found for the corresponding vertex guard problem on the same input polygon. Likewise, for MINIMUM POINT (FIXED HEIGHT) GUARD ON TERRAIN, we solve the corresponding problems with triangle restriction on the same input terrain.

In a third approach for solving MINIMUM POINT GUARD ON POLYGON we lay a grid of polynomial

[3] The algorithm works for the fixed height guard problem as well, because we can move each guard of the solution straight up to the plane at height h. This operation increases the visibility area of the guard.

size over the polygon and then compute the area of visibility for a guard at each grid point. This can be done in polynomial time, since it corresponds to constructing convex components. Here, however, we obtain the convex components by drawing lines from each grid point through each polygon vertex. We then solve the resulting MINIMUM SET COVER instance with sets for each grid point and elements for each convex component using the standard algorithm. This approach can by used for terrains as well: We just lay a regular polynomial density grid onto the plane at height h or onto the terrain itself and construct a MINIMUM SET COVER instance for the MINIMUM FIXED HEIGHT (POINT) GUARD ON TERRAIN problem. Despite of all this extra effort, the approximation ratios remain $O(n)$ in all cases, to the best of our knowledge.

5. Conclusion

We have presented approximation algorithms that achieve logarithmic approximation ratios for the problems MINIMUM VERTEX GUARD ON TERRAIN (WITH TRIANGLE RESTRICTION) and MINIMUM FIXED HEIGHT/POINT GUARD ON TERRAIN WITH TRIANGLE RESTRICTION. Furthermore, we have outlined a few heuristic approaches to solving MINIMUM FIXED HEIGHT/POINT GUARD ON TERRAIN and MINIMUM POINT GUARD ON POLYGON. It is of course of great interest and an important open problem whether heuristic approaches are all we can do to solve MINIMUM FIXED HEIGHT/POINT GUARD ON TERRAIN or whether there exist approximation algorithms that achieve approximation ratios of $o(n)$ for these problems.

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